

Comment on “Theorem for nonrotating singularity-free universes”

A.K. Raychaudhuri presented recently [1] a general theorem stating that “in any singularity-free nonrotating universe, open in all directions, the spacetime average of all stress energy invariants including the energy density vanish”. The strong energy condition is assumed. The proof was obtained by taking the spacetime average of both sides of the Raychaudhuri equation

$$\dot{\theta} - \nabla_a \dot{\xi}^a = -\sigma_{ab}\sigma^{ab} - \frac{1}{3}\theta^2 + \omega_{ab}\omega^{ab} - R_{ab}\xi^a\xi^b, \quad (1)$$

where ξ^a is the unit time-like vector along the world lines of matter. For nonrotating universes $\omega^{ab} = 0$, and with the assumption of the strong energy condition each term in the R.H.S. of (1) is non-positive. By using general arguments, he could show that $\langle \dot{\theta} \rangle = \langle \nabla_a \dot{\xi}^a \rangle = 0$, implying that the spacetime average of each term in the R.H.S. of (1) vanishes. According to the Letter, this result suggests that one should give up of a realistic singularity-free description of the universe.

We stress here that, even for realistic models with a initial singularity, the spacetime average of the invariants in the R.H.S. of (1) can vanish, and hence, this cannot be used to rule out singularity-free models. Let us take, for instance, the Robertson-Walker line element with $\kappa = 0$, $ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$. With $\xi^a = (1, 0, 0, 0)$, we have $\dot{\xi}^a = 0$ identically and

$$\begin{aligned} \langle \dot{\theta} \rangle &= \lim_{\substack{t_1, x_1, y_1, z_1 \rightarrow \infty \\ t_0 \rightarrow 0}} \frac{\int_{t_0}^{t_1} \int_{-x_1}^{x_1} \int_{-y_1}^{y_1} \int_{-z_1}^{z_1} (\xi^a \nabla_a \theta) \sqrt{g} d^4x}{\int_{t_0}^{t_1} \int_{-x_1}^{x_1} \int_{-y_1}^{y_1} \int_{-z_1}^{z_1} \sqrt{g} d^4x} \\ &= \lim_{\substack{t_1 \rightarrow \infty \\ t_0 \rightarrow 0}} 3 \frac{\int_{t_0}^{t_1} a^2 \left(\ddot{a} - \frac{\dot{a}^2}{a} \right) dt}{\int_{t_0}^{t_1} a^3 dt}. \end{aligned} \quad (2)$$

For both realistic cases of a dust-filled universe ($a(t) = Ct^{2/3}$) and a radiation-filled universe ($a(t) = Ct^{1/2}$), $\langle \dot{\theta} \rangle$ vanishes. Indeed, one has

$$\langle \dot{\theta} \rangle_{\text{rad}} = \lim_{\substack{t_1 \rightarrow \infty \\ t_0 \rightarrow 0}} -\frac{3}{2} \frac{\int_{t_0}^{t_1} t^{-1/2} dt}{\int_{t_0}^{t_1} t^{3/2} dt} = 0, \quad (3)$$

and

$$\langle \dot{\theta} \rangle_{\text{dust}} = \lim_{\substack{t_1 \rightarrow \infty \\ t_0 \rightarrow 0}} -2 \frac{\int_{t_0}^{t_1} dt}{\int_{t_0}^{t_1} t^2 dt} = 0. \quad (4)$$

As we can see, for both cases, the spacetime averages of the quantities in the R.H.S. of (1), including the energy density average $\langle (T_{ab} - \frac{1}{2}g_{ab}T) \xi^a \xi^b \rangle$, vanish.

For realistic closed Robertson-Walker universes ($\kappa = 1$), we do have finite spacetime averages. Nevertheless, as it was noted above, one cannot use the vanishing of the stress energy invariants to decide between singularity-free and singular descriptions of the universe.

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[1] A.K. Raychaudhuri, Phys. Rev. Lett. **80**, 654 (1998).